THE
MATHEMATICAL GAZETTE.

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LOCAL BRANCHES.

SOUTHAMPTON AND DISTRICT MATHEMATICAL SOCIETY.

THE Inaugural Meeting of the above Society was held at Taunton School Southampton, on Friday, October 22nd, by kind invitation of the Headmaster, Mr. S. J. Gubb, when seventeen members were present.

The report of the provisional committee was presented and discussed, and it was resolved to apply to the Mathematical Association for affiliation on the terms suggested by the Council of that Society, and to ask the Council to supply at least one copy of the *Gazette* for the use of Associate members. The opinion was also expressed that the branch (if affiliated) should be represented on the Central Council.

The following rules were passed:

(i) That this Society be called "The Southampton and District Mathematical Society."

(ii) That its object be to provide for the interchange of ideas on the teaching of mathematics.

(iii) That all persons interested in the teaching of mathematics be eligible for membership. New members may be proposed, seconded, and elected by a show of hands at any meeting.

(iv) That the subscription be 2s. 6d. per annum.

(v) That the management of the Society be in the hands of a committee consisting of (1) a President, (2) a Secretary-Treasurer, (3) three other members, at least one of whom shall, if possible, be elected from members not residing in the borough.

(vi) That there shall be at least four meetings in each session. It was also resolved that the session shall commence on January 1st.

The following officers were elected: *President*, Prof. E. L. Watkin, Hartley University College; *Secretary-Treasurer*, C. H. Holmes, Esq., Southampton Grammar School; *Committee*—Dr. Fenwick, Bournemouth Grammar School; Miss E. Hall, Southampton Girls' Grammar School; H. Wilde, Esq., Taunton Grammar School, Southampton.

A paper was then read by Prof. Watkin on "Ancient and Modern Methods in Geometry," giving an account of the position of geometry as compared with the other branches of mathematics at the time of Euclid, and emphasising the fundamental differences in the conceptions of geometry at that time as compared with the views held at present. The paper then went on

to deal with the essential features of the modern methods of teaching the subject.

On the subject being thrown open for discussion the following members spoke: Dr. Fenwick, Mr. Durrell, Mr. Tregear, Mr. Holmes.

Prof. Watkin having replied, the meeting terminated with a vote of thanks to Mr. Gubb for his hospitality.

NORTH WALES LOCAL BRANCH.

A MEETING of the North Wales branch was held on November 20th at the Girls' County School, Bangor. The proposed rules for local branches were read, and their consideration, as far as this branch is concerned, was postponed until after the general meeting of the Association.

Mr. Ferguson, Lecturer in Physics at Bangor University College, then continued the sketch (which he had begun in May) of the progress of Mathematical Analysis from ancient times. He described the success in the sixteenth century of Cardan and his pupil Ferrari in solving biquadratics, and Vieta's methods, afterwards systematised by Newton, of approximating to the roots of equations. Instances were given of the crude symbolism used by Cardan, Vieta, Montanus, and Descartes. The lecturer then referred to the invention of logarithms by Napier in the sixteenth century, before indices were used, and to the accuracy, even at that time, of astronomical observations and trigonometrical tables. The early beginnings of the Calculus were described, including Cavalieri's method of indivisibles, Newton's doctrine of Fluxions, and the use of Infinitesimals by Leibniz, as well as the controversy between these two famous men, which separated English and Continental mathematicians for more than a hundred years.

A brief reference was made to a recently proposed method of solving equations by the aid of electricity, which gives the imaginary as well as the real roots.

The next meeting will be held in February, when Prof. Bryan will read a paper on Examinations.

ON THE CONSTRUCTIONS WHICH ARE POSSIBLE BY EUCLID'S METHODS.*

§ 1. That some advantages have been derived from the freedom now enjoyed in the teaching of Geometry will be denied by no one; but there are disadvantages equally obvious. The confusion both of order and method, the neglect of the theoretical and deductive work as compared with the

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* The only reference which I can find in the English text-books to this subject is a short paragraph in *Hardy's Pure Mathematics* (pp. 64-5).

Klein's Vortrage über ausgewählte Fragen der Elementar-Geometrie (Leipzig, 1898) is out of print, and I have been unable to obtain a copy; but it is worthy of note that no one has done more for the improvement of the teaching of Elementary Mathematics in Germany than Klein, himself the most famous of living German mathematicians.

In *Weber-Wellstein's Encyklopädie der Elementar-Mathematik* (Leipzig, 1905) the teacher will find a most valuable handbook. The articles on *Kreisteilung* (Bd. I. §§ 105-7) and *Unmöglichkeitsbeweise* (Bd. I. §§ 108-114) discuss at some length the subjects to which I shall refer.

Further, any one who can read Italian will find a complete and beautiful study of these and other questions of Elementary Geometry in the collection of monographs contained in Enriques' volume, entitled *Questioni riguardanti la Geometria Elementare* (Bologna, 1900). A German translation of this work is in progress, of which the second volume has been issued, under the title, *Enriques, Fragen der Elementar-Geometrie* (Leipzig, 1907). The first volume is advertised to appear in 1910.

To the various papers in Enriques' volume I am particularly indebted, and if it had been available in English, these pages would not have been written.

practical and experimental, and the failure to grasp the value of Euclid's work as an educational discipline may be mentioned. Perhaps it is not too late to express the hope that one of our great mathematicians—or a representative group chosen from among them—may yet produce a Text-book of Geometry which will be to English-speaking people what Legendre's *Eléments* has been, during more than 100 years, to so large a part of the Continent of Europe.

Some of the methods now in use in the teaching of Geometry are also responsible for mistaken views of the nature and possibility of geometrical proof. Mistakes of this kind have even found their way into the text-books. The treatment of Parallels occasionally suggests an ignorance of the Theory of Parallels,* and the language used with regard to the construction of regular polygons, or the division of angles, might lead one to infer that such constructions were always possible. Perhaps it is partly on that account that within the last few years many so-called proofs of the Classical Problems of Elementary Geometry—known to be insoluble—have reached me from different quarters of Australia. In some cases it has been easy to detect the fallacy; in others it has been equally difficult; but in all it has been no satisfaction to the discoverers to be told that their solutions must be wrong, because it was possible to demonstrate that no such solution could be found. Their reply was always the same. You say it is impossible. Then you must be wrong. Here is the solution.

The blame rests upon our own shoulders. In our Universities, courses on the History or Principles of Mathematics are almost unknown, while in the training of every teacher of Mathematics some place should be found for the study of the History of Elementary Mathematics, and, more especially, of Elementary Geometry. The German plan † of a general course at the end of the student's third year of study seems an admirable one for the advanced student, and might be extended even to the case of those who do not pursue their studies so far. In such a course the student would learn, among other things, that Euclid's Geometry rests upon a number of fundamental hypothesis of which the Parallel Axiom (or Postulate) is one. He would learn something of the various attempts made to prove this hypothesis, and of the final discovery of the impossibility of demonstrating it. He would understand the practical reasons for accepting the Euclidean Geometry, and he would cease to believe those who have been so ready to speak lightly of the genius which created and inspired that system. He would also learn the meaning of the statement that it is impossible by Euclid's methods to

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* For the Theory of Parallels, Engel und Stäckel's *Theorie der Parallel-Linien von Euclid bis auf Gauss* (Leipzig, 1895) is the standard book; but for an elementary historical treatment of this theory, and of the rise of the different Non-Euclidean Geometries, *Bonola's La Geometria non Euclidea* (Bologna, 1906) is unique. A German translation by Liebmann of this little volume has already appeared; a Russian translation is in progress, and an English translation, which Professor Bonola has kindly permitted me to undertake, is now in the press.

† Cf. *Bull. Amer. Math. Soc.*, vol. xv., pp. 261-4 (1909). From a review of the report of the Commission appointed in 1904 by the Society of German Natural Scientists and Physicists to examine and report upon various proposed reforms in the teaching of mathematics and the natural sciences in Germany, we quote the following:

"The Commission also recommend emphatically that at the end of the general studies in pure mathematics a course be given organizing the entire mathematical material according to its essential inter-relation, and as far as possible presenting the import of the higher branches for the different stages of school mathematics. For, in fact, experience teaches that without such a course of study, the majority of the students do not discover the inner bond that connects the various parts of mathematical science, and thus the prospective teacher loses what should be for him the real gist of his mathematical studies. To avoid misunderstanding, we add expressly that this course presupposes matured hearers, and should not be brought down to the level of those preparing to teach mathematics as a minor subject only."

trisect an angle, in general, or to construct a regular polygon of n sides, unless n happens to be a number of a certain kind. He would learn the characteristic property of all constructions which are possible by Euclid's methods, and he would find that some of those which are impossible, when only the ruler and compass can be used, become possible when other instruments are admitted. He might also be shown something of the true nature of the problem of "squaring the circle," and though the complete proof of the impossibility of doing this would be too difficult, he could learn something of the nature of the proof, and the reasoning on which it is based.

All these questions must, at some time or other, have been put to most teachers by their pupils, and in many cases the answers they have given must have been sufficient to show them their own ignorance of the subject.

§ 2. In this paper I propose to discuss the question of the constructions which are possible by Euclid's methods; that is, with the ruler and compass only. In his plane Geometry points, lines, and circles only are used, and therefore these two instruments alone are at our disposal.

We start with a segment, which is taken as the unit of length, one of its ends as the origin, and the line of which it forms a part as the axis of x. The axis of y can then be constructed, and Euclid's methods allow us to obtain the position of any point in the plane of xy, whose coordinates are rational numbers.

They also allow us to obtain any point whose coordinates involve quadratic surds only.

To prove this, consider any rational number n, not a perfect square. To find $\sqrt{n}$ we need only find the mean proportional to the segments whose lengths are unity and n. This is obtained by a simple construction with the aid of the ruler and compass only.

Then we can find the segment

$$l+m\sqrt{n}$$

at once; and so we can proceed to

$$\sqrt{l+m\sqrt{n}},$$

and then to any expression, however complicated, provided the only surds it contains are square roots, repeated a finite number of times.

Further, the coordinates of all the points in the plane of xy, which can be found by Euclid's methods, can involve only irrationals of this kind.

To prove this, we must remember that we start with a unit segment only. From this we can obtain the segment representing any rational number. Now any other point—not an arbitrary point—introduced into the figure is introduced by the intersection of two straight lines, a line and a circle, or two circles. These lines and circles are definite lines and circles, and the constants in their equations

$$lx+my+n=0,$$

$$x^2+y^2+ax+by+c=0,$$

are all rational numbers.

Hence the coordinates of the points we reach in this way will take the form

$$p+q\sqrt{r}$$

where p, q, and r are rational.

Any two points thus obtained may be joined, and this segment is given by

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

This will also be an expression involving quadratic surds only, possibly repeated twice.

With the segments thus obtained we can find, as before, other straight lines and circles. The coefficients in their equations may involve quadratic

surds; but, proceeding in this way, the coordinates of the points obtained from their intersection can involve quadratic surds only, though these may take extremely complicated forms. And however far these constructions are pushed, this would still remain the case.

Hence, starting from a unit segment, Euclid's methods will give the segments whose lengths are any rational expressions, or any expressions involving quadratic surds; and they will give no others. Also they will give any point in the plane of xy, whose coordinates are rational, or involve quadratic surds only; and they will give no other points. Neither would the introduction of Solid Geometry give other segments or points in this plane, for the equations of the plane and sphere, being of the first and second degree, the coordinates obtained could involve quadratic surds only.

§ **3.** Now whatever be the problem to be solved, the data may be taken as points, limited in number. Two of these points being joined, this line may be taken as the axis of x, the segment as the unit segment, and one of its ends as the origin. The axis of y can then be constructed, and the coordinates of the other points are the additional data of the problem.

If these coordinates are either rational or involve quadratic surds, then the argument of the preceding article shows that the only points we can obtain in the plane, by Euclid's methods, must have for their coordinates, rational expressions, or expressions involving quadratic surds; and that we can obtain any point whose coordinates are of this nature.

If among the data are other irrationalities, the construction could lead to points whose coordinates might involve these irrationalities, or any combination of the data, obtained by carrying out rational operations, or successive extraction of square roots upon them.

The arbitrary points introduced into the construction being arbitrary, their exact position must not affect the final answer. If they are wholly arbitrary, then their coordinates may be taken as rational. If they are to lie on certain fixed lines or circles, then these lines and circles are obtained by Euclid's methods, and the coordinates of the points may be taken as given in the above manner. If they are to lie on arbitrary lines or circles, then these may be taken as given in this way. It is thus clear that by the introduction into the figure of arbitrary points—and this includes also arbitrary lines and circles, as these are determined by points—the coordinates finally obtained in the construction do not pass outside the range above referred to.

If the given segments are 1, a, b, c, etc., we could obtain, by the ruler alone, any segment in the domain of rationality, $R\,(1, a, b, c \dots)$. With both the ruler and compass we could obtain any segment in the domain of rationality which might be denoted by $R^{\frac{1}{2}}$.

We may state the result we have now established in the following words:

Let a geometrical problem be proposed for solution, reducing to the determination of points in a plane, which are related to certain definite points in that plane by some given conditions. In order that the required points may be constructed by the ruler and compass only (and with the aid of the given points, and other arbitrary points, lines, or circles, if desired), it is necessary and sufficient that the coordinates of the required points may be obtained by carrying out rational operations, or successive extractions of square roots, finite in number, on the given coordinates.

§ 4. **The Duplication of the Cube.**

We shall now examine several of the Classical Problems of Elementary Geometry, which can be shown to be insoluble, by Euclid's methods, from the theorem we have just enunciated. First we take the case of the Duplication of the Cube, one of the most ancient of these problems, and also, from the algebraical point of view, one of the simplest.

It is required to construct a cube which shall be double a given cube.

Let the side of the given cube be taken as the unit segment.

Let the side of the required cube be x units.

Then we must have
$$x^3-2=0.$$

But the only real root of this equation is $\sqrt[3]{2}$; and we have seen above that, by the aid of the ruler and compass alone, we cannot obtain a segment whose length is a cube root (irrational).

Therefore the Duplication of the Cube is impossible by Euclid's methods.

§ 5. **The Regular Heptagon.**

It is required to inscribe a regular heptagon in a given circle.

Let the centre of the circle be taken as origin, and its radius as the unit of length.

Suppose the angular points are A_0, A_1, ... A_6, and that OA_0 lies along the axis of x.

Then the coordinates (x_1, y_1) of A_1 are $\cos\frac{2\pi}{7}$ and $\sin\frac{2\pi}{7}$.

The coordinates (x_2, y_2) of A_2 are $\cos\frac{4\pi}{7}$ and $\sin\frac{4\pi}{7}$, etc.

Put
$$z_r=x_r+iy_r.$$

Then z_1, z_2, z_3 ... z_6 are the roots of the equation
$$\frac{z^7-1}{z-1}=0.$$

Thus if the coordinates (x_1, y_1), etc., are rational, or involve quadratic surds only, the roots of this equation must be rational, or involve quadratic surds only; and if the roots of this equation are of that nature, the coordinates of the angular points can be obtained with the ruler and compass only.

We have thus to consider the equation
$$z^6+z^5+z^4+z^3+z^2+z+1=0.$$

On substituting
$$z+\frac{1}{z}=u,$$
we have
$$u^3+u^2-2u-1=0.$$

Obviously the roots of this equation must all be real, since they are equal to $2x_1$, $2x_2$ and $2x_3$.

Further, they cannot be integral, since the only possible integers would be ± 1, and they do not satisfy the equation.

Also they cannot be rational fractions, since if the rational fraction $\frac{m}{n}$, in its lowest terms, were a root, we would have
$$\frac{m^3}{n}+2m^2-2mn-n^2=0\,;$$
and this is impossible.

Thus the roots of the equation
$$u^3+u^2-2u-1=0$$
are all irrational.

But they cannot involve quadratic surds only, since the degree of any irreducible equation, with rational coefficients, whose roots involve quadratic surds only, must be a power of 2.

This will be almost evident from an example.

Let
$$x=\sqrt{1+\sqrt{2}}.$$
Then we obtain
$$x^2=1+\sqrt{2},$$
and finally
$$(x^2-1)^2=2,$$

and this equation, whose coefficients are rational, is irreducible, and of the fourth degree. Indeed the roots of this equation are the four surds

$$\pm\sqrt{1\pm\sqrt{2}}.$$

For a rigorous proof of this property cf. *Weber-Wellstein, loc. cit.*, Bd. I., §§ 108-9; *Enriques, loc. cit.*, p. 363, *et seq.*; *Petersen, Théorie des Equations Algébriques*, p. 150 (*Paris*, 1897).

Since these roots cannot involve quadratic surds only, and since they are equal to $2x_1$, $2x_2$ and $2x_3$, the angular points of the polygon cannot be obtained by Euclid's methods.

Therefore it is impossible by Euclid's methods to inscribe a regular heptagon in a given circle.

§ 6. **The Regular Nonagon.**

It is required to inscribe a regular polygon of nine sides in a given circle.

Proceeding as in the last article, the coordinates of the angular points are given by the real and imaginary parts of the roots of the equation

$$\frac{z^9-1}{z-1}=0;$$

that is, of the equation

$$(z^2+z+1)(z^6+z^3+1)=0.$$

Remembering that

$$z_r=\cos\frac{2r\pi}{9}+i\sin\frac{2r\pi}{9}$$

we see that

$$z^2+z+1=0$$

gives

$$z_3+\frac{1}{z_3}=-1=2x_3.$$

Also the equation

$$z^6+z^3+1=0$$

reduces to

$$u^3-3u+1=0,$$

on putting

$$z+\frac{1}{z}=u.$$

The roots of this equation are all real, and they cannot be either integers or rational fractions.

Therefore they must be irrational, and, as the equation is of the third degree, they cannot involve quadratic surds.

Therefore

$$z_1+\frac{1}{z_1}=2x_1,$$

$$z_2+\frac{1}{z_2}=2x_2,$$

$$z_4+\frac{1}{z_4}=2x_4,$$

do not involve quadratic surds.

Thus, starting with A_0, we can find A_3 and A_6 by Euclid's methods, but we cannot obtain the points A_1, A_2 and A_4, or A_5, A_7 and A_8.

Therefore it is impossible to inscribe a regular nonagon in a given circle by Euclid's methods.

§ 7. **The Trisection of an Angle.**

The last three problems have all reduced to the solution of cubic equations where roots could not be obtained by Euclid's methods. The Trisection of an Angle also reduces to the solution of a cubic, but in many cases this equation can be solved rationally, so that in these cases the

angle can be trisected. Indeed, in Elementary Geometry we are shown how to trisect a right angle; *and when this is done, and the angle* $\frac{2\pi}{n}$ *is given, if* n *is not a multiple of three, the angle* $\frac{2\pi}{n}$ *can be trisected.*

This can be shown as follows:

Let
$$n=3p\pm 1.$$

Then we have
$$\frac{2\pi}{3}-p\left(\frac{2\pi}{3p+1}\right)=\frac{2\pi}{3(3p+1)}$$
and
$$p\left(\frac{2\pi}{3p-1}\right)-\frac{2\pi}{3}=\frac{2\pi}{3(3p-1)}.$$

Thus in these cases the angle can be trisected.

We shall see later, in dealing with the regular polygons, that the angle $\frac{2\pi}{n}$ can be constructed only when n, expressed in its prime factors, takes the form
$$2^r(2^{2^{r_1}}+1)(2^{2^{r_2}}+1)\ldots(2^{2^{r_s}}+1),$$
$r_1, r_2 \ldots r_s$ being all different positive integers.

Therefore, if, in addition, n is not a multiple of three, the angle $\frac{2\pi}{n}$ can be trisected.

We proceed to the analytical discussion of the problem.

Let the angle θ be the given angle.

Then we are given the trigonometrical ratios of the angle.

In particular we are given $\cos\theta$, and we know that
$$4\cos^3\frac{\theta}{3}-3\cos\frac{\theta}{3}=\cos\theta.$$

If the values of $\cos\frac{\theta}{3}$ can be obtained in terms of $\cos\theta$ by rational operations, or operations involving extraction of square roots only, then $\cos\frac{\theta}{3}$ can be found by the ruler and compass, and the angle $\frac{\theta}{3}$ can be constructed.

Conversely, if the angle θ can be trisected, $\cos\frac{\theta}{3}$ must be given in this way.

Put
$$\cos\theta=a \text{ and } 2\cos\frac{\theta}{3}=x.$$

Then we have
$$x^3-3x=2a.$$

For an infinite number of values of a, this equation can be solved rationally in terms of a, or by successive extractions of square roots. To construct such cases, we need only take any segment a, where $|a|<2$, which can be constructed by Euclid's methods, and put
$$a^3-3a=2a.$$

Then the angle $\cos^{-1}a$ can be trisected.

We shall now show that, in general, it is impossible to trisect a given angle.

To prove this we need only show that the equation
$$x^3-3x=2a,$$
will not, in general, be soluble rationally in terms of a, or by operations involving successive extraction of square roots.

Now if the equation could be solved in this way, one of the roots must be rational, since three irrational roots of this kind could not give rise to this cubic.

However, it is easy to manufacture such equations for which no rational root is possible. One such case is given by $\theta=\frac{\pi}{3}$.

Then $$\cos\theta=-\tfrac{1}{2},$$
and the equation is $$x^3-3x+1=0. \qquad \text{(cf. § 6).}$$

Also there are an infinite number of such cases.

For putting $$\cos\theta=\frac{m}{n}, \text{ and } nx=y,$$
we have $$y^3-3n^2y=2mn^2.$$

If m, n are positive integers, the rational roots of this equation must be integral.

Let the fraction $\frac{m}{n}$ be in its lowest terms, and n an odd number whose prime factors are all different: *e.g.* $n=p\,.\,q$.

Then, since $$y^2-3n^2=\frac{2mn^2}{y}$$
the possible factors of y are 2, in p or q.

But p cannot be a factor of y, since in that case the left hand side of the equation $$y^3-3n^2y=2mn^2$$
would contain p^3, and the right hand side would contain only p^2. Similarly for the other factors of n.

Also it is clear that m cannot be a factor, and that, in general, y cannot be equal to 2.

Thus there are an infinite number of cases in which the cubic equation $$x^3-3x=2a$$
cannot be solved by rational operations, or operations involving successive extraction of square roots.

*Hence any geometrical solution—with the ruler and compass only—which pretends to give the Trisection of any given Angle must be fallacious.**

§ 8. Another proof of this result is as follows:

Let $$u=\cos\theta, \text{ and } v=\sin\theta \text{ be given.}$$
Then $$u+iv=\cos\theta+i\sin\theta=\lambda, \text{ say.}$$
Put $$x=\cos\frac{\theta}{3} \text{ and } y=\sin\frac{\theta}{3},$$
and let $$x+iy=z.$$
Then we have $$z^3-\lambda=0.$$

Thus the trisection of the angle will be possible, if the equation $$z^3-\lambda=0$$
can be solved by rational operations, or operations involving successive extraction of square roots, on the data 1, λ; and, if the trisection of the angle is possible, z must be of this form.

Hence we have to show that, in general, $$z^3-\lambda$$
cannot be replaced by $$[z-f(\lambda)][z^2+g(\lambda)z+h(\lambda)]$$
where f, g, h, are rational functions of λ.

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* For the Trisection of an Angle by means of Conics, or other curves, or by mechanical means, cf. *Enriques, loc. cit.*, p. 451, *et seq.*

But, if the analytic functions $\sqrt[3]{\lambda}$ and $f(\lambda)$,

of the complex variable λ were to be equal for any finite portion of the circumference of the unit circle whose centre is at the origin, we know from the Theory of Functions of a Complex Variable that they would need to be equal for all values of the variable.

This would require, among other things, that for all real values of λ

$$\sqrt[3]{\lambda}=f(\lambda),$$

where $f(\lambda)$ is some rational function of λ; and this is impossible.

Therefore we have proved that the equation

$$z^3-\lambda=0$$

*is, in general, irreducible, and that the Trisection of an Angle is, in general, impossible by Euclid's methods.**

§ 9. These examples, and the discussion in the preceding articles, will have shown that by the introduction of the methods of Analytical Geometry the possibility or impossibility of performing certain constructions by Euclid's methods was most likely to be established after, and owing to, a fuller study of the theory of algebraical equations. In this study the great German mathematician, Gauss (1777-1855), led the way, and his discoveries on the nature of the roots of certain algebraical equations enabled him to decide what regular polygons could be constructed by Euclid's methods. The other two mathematicians whose work in the theory of algebraical equations marked an epoch in the development of Pure Mathematics were Abel (1802-1829) and Galois (1811-1832). They both died in early youth. Yet their work influenced the advance of mathematics right through the nineteenth century.

In all the cases to which we have referred above, the geometrical relations could be translated without difficulty into algebraical equations, and the discussion of the nature of the roots of these equations decided the question of the possibility or impossibility of the construction. There remains one famous problem—the question whether it is possible to construct by Euclid's methods a square equal in area to a given circle—in which the first difficulty consisted in finding an algebraical statement of the problem. Since the area of a circle of radius r is equal to πr^2, the circle is equal to a rectangle, one of whose sides is equal to the radius, and the other to half the circumference. If a square could be constructed equal to the circle, then this rectangle could be constructed: and, taking the radius of the circle for the unit segment, one of the sides of the rectangle would be equal to π, and this segment could be constructed.

It had long been known that π was irrational. However, it was only in 1882 that it was definitely established that it was transcendental; that is, that it could not be the root of any algebraical equation with rational coefficients.†

The proof of this theorem not only established the fact that the circle cannot be squared by Euclid's methods (since in that case π would have been the root of an algebraical equation whose degree was a power of 2), but it also showed that such a construction could not be made, even if algebraical curves of any order were allowed. The problem of "squaring the circle" was thus finally placed among those which are impossible of solution.

H. S. Carslaw, Sc.D.

Sydney, N.S.W.

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* Cf. Forsyth's *Theory of Functions*, ch. iv., § 37.

† This discovery was made by Lindemann: cf. *Math. Ann.*, Bd. XX., p. 213. See *Klein, Lectures on Elementary Mathematics* (*The Evanston Colloquium*), p. 51 (1894), and *Enriques, loc. cit.*, p. 471, *et seq.*

AN EASY INTRODUCTION TO THE NATURAL BASE OF LOGARITHMS.

I. Consider the two curves $[y=a^x]$ and $[y=b^x]$, where a and b are each >1. Let a straight line through the origin O meet the first curve in C and D, and let straight lines through C and D drawn parallel to the x-axis meet

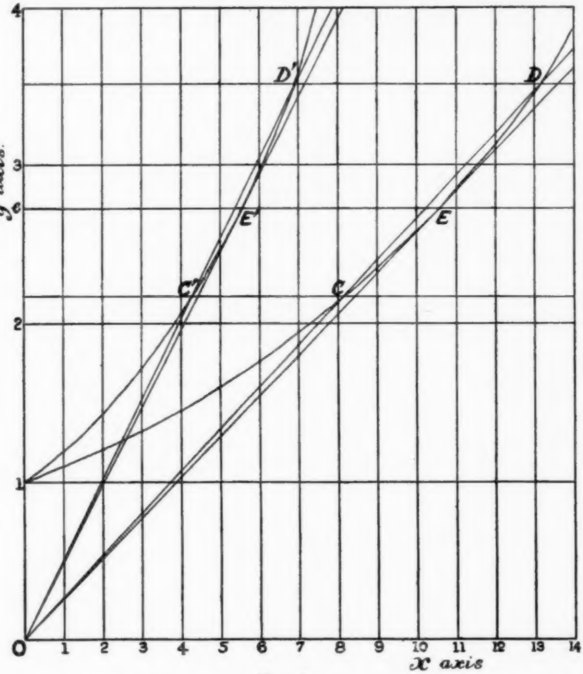


FIG. 1.

the curve $[y=b^x]$ in C' and D' respectively. [See Fig. 1.] Then, denoting the x of the point C by x_C, etc., we have

$$a^{x_C}=b^{x_{C'}};$$

$$\therefore\ x_C \,.\, \log a = x_{C'} \,.\, \log b;$$

$$\therefore\ x_C/x_{C'} = \log b/\log a;$$

Similarly, $$x_D/x_{D'} = \log b/\log a;$$

$$\therefore\ x_C/x_{C'} = x_D/x_{D'}.$$

This shows that the points O, C', D' are in one straight line.

Now let a tangent from O touch the first curve at E. When the points C and D coincide at E, so that OCD becomes the tangent OE, the points C' and D' also coincide, so that $OC'D'$ becomes the tangent at E', and the points E and E' have equal ordinates.

Hence, if a is any constant greater than 1, and if a straight line OE is drawn from the origin O to touch the curve $[y=a^x]$ in the point E, then the value of the ordinate of E is independent of a,

i.e. is an absolute constant. ....(1)

The two curves of Fig. 1 are the graphs of $y=(1\cdot1)^x$ and $y=(1\cdot2)^x$, and the diagram suggests some suitable exercises for the beginner to determine approximately the value of this constant ordinate.

II. Let (x, y) be the co-ordinates of any point P on the curve $[y=a^x]$, and let x receive a small increment h, so that

$$\Delta x = h.$$

Then
$$\Delta y = a^{x+h} - a^x$$
$$= a^x(a^h - 1)\,;$$
$$\therefore\ \frac{\Delta y}{\Delta x} = y\left(\frac{a^h-1}{h}\right);$$
$$\therefore\ \left.\begin{aligned}\frac{dy}{dx} &= y \times \operatorname*{Lt}_{h\to 0}\left(\frac{a^h-1}{h}\right)\\ &= y\,.\,f(a)\end{aligned}\right\} \qquad \text{..........................(2)}$$

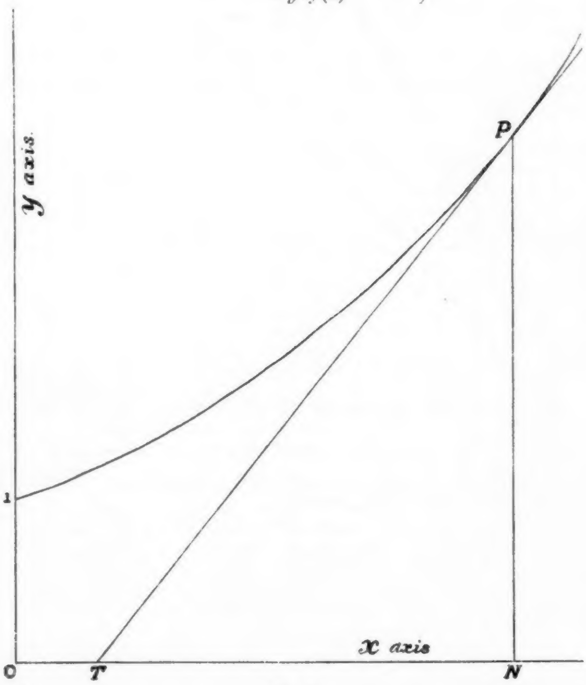


Fig. 2.

The function $f(a)$ involves a only, and not the position of the point P on the curve $[y=a^x]$. If the tangent at P meets the x-axis in T (Fig. 2),

the gradient of the tangent TP is y/TN. Hence the subtangent TN is constant, being always $1/f(a)$ for all positions of the point P on the curve.

III. Now let the absolutely constant value of the ordinate of E be denoted by e, and let e be taken as the base of logarithms.

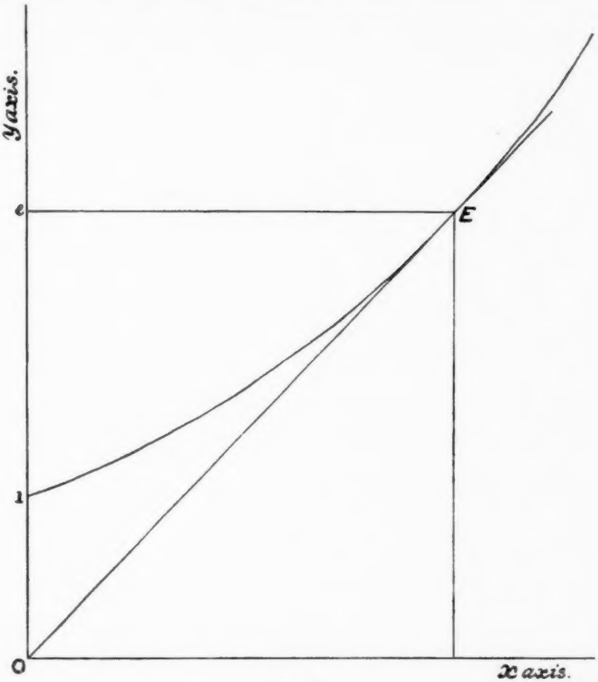


FIG. 3.

Applying the result (2) to the case where the point P is taken at E, so that OE is the tangent at E (Fig. 3), we have

$$e/x_E = e \,.\, f(a)\,;$$

$$\therefore \; x_E = 1/f(a).$$

But

$$e = a^{x_E}\,;$$

$$\therefore \; e = a^{1/f(a)}\,;$$

$$\therefore \; e^{f(a)} = a,$$

$$\left.\begin{array}{r} i.e. \; f(a) \equiv \log(a), \\ i.e. \; \underset{h\to 0}{\mathrm{Lt}} \left(\dfrac{a^h - 1}{h}\right) = \log a. \end{array}\right\} \quad \text{.................................(3)}$$

As a particular case, writing e for a, we have

$$\underset{h\to 0}{\mathrm{Lt}} \left(\dfrac{e^h - 1}{h}\right) = 1.$$

Now put $h=1/n$, so that n is ∞ when h is 0; thus we see that

$$\underset{n\to\infty}{\mathrm{Lt}}\,[n(e^{1/n}-1)]=1\,;$$

$$\therefore\ \underset{n\to\infty}{\mathrm{Lt}}\,e^{1/n}=1+1/n\,;$$

$$\therefore\ e=\underset{n\to\infty}{\mathrm{Lt}}\,[(1+1/n)^n]. \qquad\ldots\ldots\ldots\ldots(4)$$

Referring again to result (2) obtained above, we have

$$\left.\begin{aligned} &da^x/dx=a^x.\log a,\\ \text{and, as a particular case,}\quad &de^x/dx=e^x.\end{aligned}\right\} \qquad\ldots\ldots\ldots\ldots(5)$$

Now let $\qquad y=\log x,$

so that $\qquad e^y=x\,;$

$$\therefore\ e^y.\,dy/dx=1\,;$$

$$\therefore\ dy/dx=1/e^y=1/x,$$

$$i.e.\ d(\log x)/dx=1/x. \qquad\ldots\ldots\ldots\ldots(6)$$

W. J. DOBBS.

SOME PROOFS OF THE THEOREM THAT A CHORD OF A CIRCLE PASSING THROUGH A POINT IS DIVIDED HARMONICALLY BY THAT POINT AND ITS POLAR, ETC.

In the proofs which follow, the definition chosen for the polar of a point P with respect to a circle of radius r and centre O is "the line perpendicular to OP which cuts it at a point P' such that $OP\,.\,OP'=r^2$." When segments of the same line are considered, regard is paid to sense as well as magnitude. As usual, two points are said to be "conjugate to one another" when each lies on the polar of the other.

First Proof.

Let P be a point on the chord AB of a circle whose centre is O, and let QR, the polar of P, meet OP in Q and AB in R, so that $O\hat{Q}R=90°$, and $OQ\,.\,OP=r^2$.

Let OM be the perpendicular from O on AB, so that M bisects AB. It suffices to prove that $MP\,.\,MR=MA^2$.

Now the circle on OR as diameter passes through M and Q,

$$\therefore\ PR\,.\,MP=OP\,.\,PQ\,;$$

$$\begin{aligned}\therefore\ MR\,.\,MP&=(MP+PR)MP=MP^2+PR\,.\,MP\\ &=OP^2-OM^2+OP\,.\,PQ=OP(OP+PQ)-OM^2\\ &=OP\,.\,OQ-OM^2=r^2-OM^2=MA^2. \qquad \text{Q.E.D.}\end{aligned}$$

Second Proof.

Let O be the centre of a circle, of which CD is a diameter passing through P, and let AB be a chord passing through P.

Complete the quadrilateral PAB, PCD, AC, BD; and let AC, BD meet in S and AD, BC in T.

Let ST meet PAB in R and PCD in Q.

Then T is the orthocentre of the triangle SCD; $\therefore$ $SRTQ$ is $\perp^r$ to PCD. Now by property of complete quadrilateral, AB is divided harmonically by P and R, and CD by P and Q. Hence OQ, $OP=OC^2$; $QTRS$ is polar of P.

Thus AB is divided harmonically by P and its polar.

Third Proof.

Joining O and Q to A and B in the figure of the 1st Proof,

$$\frac{OQ}{OA}=\frac{OA}{OP}; \therefore OQA,\ OAP \text{ are similar triangles};$$

$$\therefore A\hat{Q}P=O\hat{A}B,$$

$$\frac{OQ}{OB}=\frac{OB}{OP}; \therefore OQB,\ OBP \text{ are similar triangles};$$

$$\therefore O\hat{Q}B=O\hat{B}P=O\hat{B}A=O\hat{A}B=A\hat{Q}P;$$

$\therefore$ OQP is exterior or interior bisector of $A\hat{Q}B$;
$\therefore$ QR is interior or exterior „ „
$\therefore$ P and R divide AB harmonically.

Fourth Proof.

To deduce the Theorem from the proposition that the square on the distance between two 'conjugate' points equals the sum of the 'powers' of these points as to the circle.

$$PR^2=PA\,.\,PB+RA\,.\,RB;$$

$$\therefore (PA+AR)(PB+BR)=PA\,.\,PB+RA\,.\,RB;$$

$$\therefore PA\,.\,PB+PA\,.\,BR+AR\,.\,PB+AR\,.\,BR=PA\,.\,PB+AR\,.\,BR;$$

$$\therefore PA\,.\,BR+AR\,.\,PB=0;$$

$$\therefore \frac{PA}{AR}\cdot\frac{RB}{BP}=-1.$$

Q.E.D.

Conversely, from the theorem of the harmonic property of pole and polar, we can deduce the property of conjugate points.

Fifth Proof.

PS and PT are tangents drawn from P to a circle, and AB is a chord passing through P and cut by ST in R.

We have
$$\frac{PA}{PB}:\frac{RA}{RB}=\frac{\sin P\hat{T}A}{\sin P\hat{T}B}:\frac{\sin R\hat{T}A}{\sin R\hat{T}B}$$

$$=\frac{\sin R\hat{S}A}{\sin R\hat{S}B}:\frac{\sin P\hat{S}A}{\sin P\hat{S}B}$$

$$=\frac{RA}{RB}:\frac{PA}{PB};$$

$$\therefore \left\{\frac{PA}{PB}:\frac{RA}{RB}\right\}^2=1;$$

$$\therefore \frac{PA}{PB}:\frac{RA}{RB}=-1.$$

Q.E.D.

We might avoid using Trigonometry thus:

$$\frac{PA}{PB}=\frac{PA}{PT}\cdot\frac{PT}{PB}=\frac{PA}{PT}\cdot\frac{PS}{PB}=\frac{AT}{TB}\cdot\frac{AS}{SB},$$

$$\frac{RA}{RB}=\frac{RA}{RS}\cdot\frac{RS}{RB}=-\frac{AT}{SB}\cdot\frac{AS}{TB}=-\frac{PA}{PB};$$

$$\therefore \frac{PA}{PB}:\frac{RA}{RB}=-1.$$

Q.E.D.

In this form of the proof, we must fix a positive direction for each line, in order to deduce conclusively the *sign* of the double-ratio.

This method of proof suggests this interesting theorem: "If the rectangles contained by opposite sides of a cyclic quadrilateral are equal, then the tangents at its corners meet in pairs on the diagonals," which we may prove thus:

Let $ASBT$ be a cyclic quadrilateral having $AT.SB=AS.TB$. Let the tangent at T meet AB in P. Let AB and TS meet in Q. Join PS.

Then the following pairs of triangles are similar to each other: PAT, PTB; AQT, SQB; AQS, BQT.

$$\frac{AS}{SB}=\frac{AT}{TB} \text{ (since } AT.SB=AS.TB).$$

Now
$$\frac{PA}{PB}=\frac{PA}{PT}\cdot\frac{PT}{PB}=\frac{AT}{TB}\cdot\frac{AT}{TB}=\frac{AT}{TB}\cdot\frac{AS}{SB}$$
$$=\frac{AT}{SB}\cdot\frac{AS}{TB}=\frac{AQ}{QS}\cdot\frac{QS}{QB}=-\frac{QA}{QB};$$

$\therefore$ Q as well as T are on the polar of P, $\therefore$ S is on the polar of P;

$\therefore$ P is on the polar of S, *i.e.* the tangent at S passes through P.

Thus the tangents at S and T intersect on AB.

Similarly the tangents at A and B will meet at a point R in TS.

Cor. If AT and SB meet in K, and SA, BT in H, then $KATBHSA$ is a complete quadrilateral, so that P and Q divide AB harmonically, and Q and R divide ST harmonically. PQR is a self-polar triangle and K, H lie on PR, the polar of Q.

The above five proofs of the standard theorem have been chosen for their simplicity. Some will be recognized as well known. So far as I know, the 5th has not hitherto been printed. The 2nd is attributed (in *Edin. Math. Soc. Proc.*, vol. xxiv. p. 31) by Dr. P. Pinkerton to his pupil Mr. R. Vickers.

In this connection it may be worth while to call attention to certain equivalent forms of stating the condition that P and R are harmonic conjugates with respect to AB, when M is the mid point of AB.

(1) $\dfrac{PA}{AR}\cdot\dfrac{RB}{BP}=-1$ or $PA.RB+AR.BP=0$.

(2) $MR.MP=MA^2$.

(Deduce from (1) by substituting $PM+MA$ for PA, $RM+MB$ for RB, etc., and expanding, remembering that $MA+MB=0$, or $AM=MB$.)

(3) $PR^2=PA.PB+RA.RB$.

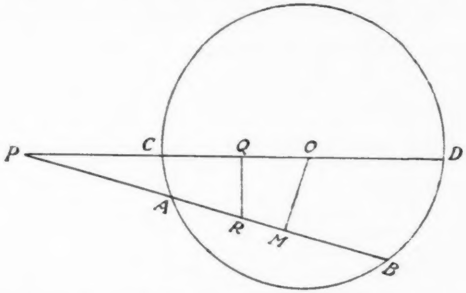
For $PR^2=(PA+AR)(PB+BR)=PA.PB+PA.BR+AR.PB+AR.BR$
$=PA.PB+RA.RB$, by (1).

(4) $PR^2=PM^2+MR^2-2AM^2=PM^2-AM^2+MR^2-AM^2$.

For $PR^2=(PM+MR)^2=PM^2+MR^2+2PM.MR$
$=PM^2+MR^2-2MA^2$ by (2).

(3) and (4) are closely connected with the theorem that the sum of the powers as to a circle of two points which are conjugate as to that circle is equal to the square on the distance between them.

If we take the figure, in which CD is a diameter passing through P, of a circle whose centre is O and AB a chord passing through P, whose mid point is M, and QR is the polar of P cutting CD in Q and AB in R, then $O\hat{Q}M = O\hat{M}A = 90^\circ$ and $OQ \,.\, OP = OC^2$.



If we substitute P, Q, C, D, O for P, R, A, B, M in the formulae (1), (2), (3), (4), then $OQ \,.\, OP = OC^2$ corresponds to (2), and formulae corresponding to (1), (3), (4) an equivalent to it. If we start from any one of these, and deduce from it any of the 4 original formulae, then we have obtained a proof of the theorem for which five proofs have already been given. Our first proof is of this type.

As another example, let us deduce (4) from the corresponding formula

$$PQ^2 = PO^2 + OQ^2 - 2OC^2.$$

We have

$$\begin{aligned} PR^2 &= PQ^2 + QR^2 \\ &= PO^2 + OQ^2 - 2OC^2 + QR^2 \\ &= PM^2 + MO^2 - 2OC^2 + OQ^2 + QR^2 \\ &= PM^2 + MO^2 - 2OC^2 + OR^2 \\ &= PM^2 + MO^2 - 2OC^2 + OM^2 + MR^2 \\ &= PM^2 + MR^2 - 2(OA^2 - OM^2) \\ &= PM^2 + MR^2 - 2AM^2. \end{aligned}$$

Q.E.D.

R. F. Muirhead.

REVIEWS.

Geometry for Beginners. By C. Godfrey, M.A., and A. W. Siddons, M.A. Pp. x + 80. Price 1s. (Cambridge Press.)

The preface states that "this book has been largely inspired by the Board of Education's Circular on the Teaching of Geometry (and Graphical Algebra) in Secondary Schools," and, further, that it consists of:

"First Stage. Introductory practical work concerned with the fundamental concepts, and not primarily designed to give facility in using instruments.

"Second Stage. Experimental work leading up to the discovery of the fundamental facts of geometry; including angles at a point, parallels, angles of a triangle and polygon, congruent triangles."

As a consequence of these aims, the book is to all intents and purposes a collection of nearly 500 questions, with a few definitions and statements of the facts which have been learned from the answers to the questions. Many of the questions are exceedingly simple verbal questions. The authors have succeeded very well in what they have set out to do. The only place in which they are liable to criticism is in their treatment of the difficult question of parallel straight

lines. The word 'parallel' first occurs on p. 12, and next in Ex. 158, which asks for instances of parallel straight lines; but the term appears to require a little more explanation in lieu of a formal definition. No doubt the intention is that the teacher should fill in such gaps as this.

The pupil who has worked through this book is in a position to take up the authors' Elementary Geometry at the section on Constructions. A special edition of the latter book is now issued without the part which this little book supersedes.

Experimental Mechanics for Schools. By F. Charles, B.A., and W. H. Hewitt, B.A., B.Sc. Pp. 288. Price 3s. 6d. (George Bell & Co.)

Here we have a book of experiments and examples interspersed with a number of theoretical proofs of certain theorems of both Statics and Dynamics. The general plan appears to be to establish experimental results when possible, and sometimes to add a theoretical proof as well. The scope of the book is wide, extending to graphical work on frames and moments of inertia.

The book gives one the impression that too much has been attempted, so that there is a lack of completeness; thus, while 'resultants' and 'equilibrants' are explained, 'moments' and 'centres of gravity' are not. A property of couples is used before the word 'couple' is defined! Unfortunately, the authors have allowed a number of serious errors to pass in the proof. For instance, the experiment in § 24 is not satisfactorily explained, and might easily, as it appears, give rise to an erroneous conclusion. The enunciation of § 52 is not quite what was intended. In § 55 four forces in equilibrium are acting on a body; it is not true that "evidently the rigid body could have been replaced by any system of rods, provided only that their extremities lay on the lines of action of the forces."

On page 33 is a list of C.G.'s, but those of a circular arc, a circular sector, and a spherical sector are wrong.

A more serious error is, however, to be found in the chapter on motion in a circle, from which it would appear that $\frac{mu^2}{r}$ is a force acting *on* the body *away* from the centre of the circle. No attempt is made to explain it as a *fictitious outward* force for the purpose of reducing the dynamical equations to statical ones. It is impossible to be too careful in dealing with the so-called centrifugal force.

The definitions arising in connection with the Laws of Motion are exceedingly vague in § 98, and in several other places the book wants most careful revision. The authors, in fact, seem sometimes to be giving a logical sequence of results of mechanics, and sometimes to be merely providing an experimental companion to such a course.

The experiments described are mostly familiar ones—Atwood's machine appears amongst them. The most attractive feature of the book is its fine collection of examples. The miscellaneous examples cover 86 pages. The last 41 examples are practical ones taken from Civil Service and L.C.C. papers.

Elementary Algebra. By A. E. Layng, M.A. Pp. 464, with Answers. Price 4s. 6d. (Blackie.)

Mr. Layng's book takes the pupil from the very beginning of the subject as far as the Experimental and Logarithmic Series, and almost the last piece of work is the method of calculating Napierian Logarithms. The book rightly follows what ought to be the universal practice of introducing simple equations at a very early stage. Factorisation is taken immediately after Multiplication, and simple identities are proved by the Remainder Theorem immediately after Division. Graphical methods of solving equations are given, and the opportunity is taken of doing some coordinate geometry of the straight line and circle. Other interesting features are the early introduction of Inequalities, Ratio, Proportion, and Variation, and the very sensible plan of giving a short chapter on Convergence, before dealing with the Binomial Theorem for any index.

The explanations are lucid throughout, as in this author's other books, and there are large numbers of examples, chiefly of the formal type; but we believe that this and most other algebras would be much improved by a larger number of more 'practical' examples in the interests of those pupils whose studies are not likely to go beyond such a book as this. In particular, such examples as the

physical ones on Variation on p. 281 should be more numerous, and the true utility of the approximation $(1+x)^n=1+nx$ is better appreciated when used in such problems as finding the loss in beats of a seconds pendulum during a day when g varies, than in merely calculating such things as $(1\cdot 04)^6$ to 3 places of decimals.

Taschenbuch für Mathematiker und Physiker. 1 Jahrgang, 1909. By F. AUERBACK, with others. Pp. 450+44 (introduction). 6 marks. (Teubner.)

This is the first publication of a reference book for mathematics and mathematical physics, which is intended to appear annually. It measures $7\frac{1}{4}$ by 5 inches, so is literally a 'pocket-book.' It is astonishing to find how much information it contains. There is a frontispiece of Lord Kelvin and an appreciation of his work by the editor. Then there follow astronomical statistics and tables of 4-figure logarithms, etc.; all this forms the introduction. The main part of the book consists of the definitions, theorems, and formulae, which are of greatest importance in nearly every branch of mathematics and physics, and in the last twenty pages even a little general chemistry is added. It is interesting to notice how little space the really important formulae in some subjects take up; thus Plane Trigonometry, adequately treated, fits into almost six pages, yet the Curvature of Surfaces requires eight. Plane Elementary Geometry appears to have suffered most in the process of condensation, and may be said to have almost shrunk out of existence.

There is an appendix containing lists of scientific periodicals, new books (in various languages), an index, and an obituary.

The authors hope to extend the scope of the book in future editions, and the indications given here and there promise that the extension may be considerable, and may, among other subjects now omitted, include Invariant Theory and Crystallography.

The book, which does not appear to contain many misprints, is well printed, and should be very useful to the competent mathematician. W. M. ROBERTS.

MATHEMATICAL NOTES.

308. [E. 5.] *On a certain form of Definite Integral.*

From the formula of reduction,

$$(m+1)\int x^m(\log x)^n\,.\,dx=x^{m+1}(\log x)^n-n\int x^m(\log x)^{n-1}\,.\,dx,$$

we get $$(m+1)\int_0^1 x^m(\log x)^n\,.\,dx=-n\int_0^1 x^m(\log x)^{n-1}\,.\,dx,$$

and hence, by a repeated application of this result, we deduce

$$\int_0^1 x^m(\log x)^n\,.\,dx=(-1)^n n!/(m+1)^{n+1}. \qquad\text{(A)}$$

Now since $$x^{cx}=e^{cx\log x},$$

expanding by the exponential theorem, and using (A), we shall get

$$\int_0^1 x^{cx}\,.\,dx=1-\frac{c}{2^2}+\frac{c^2}{3^3}-\frac{c^3}{4^4}+\dots \qquad (1)$$

Further, multiplying by x^a the series which is the expansion of x^{cx}, we shall get, in like manner,

$$\int_0^1 x^{a+cx}\,.\,dx=\frac{1}{a+1}-\frac{c}{(a+2)^2}+\frac{c^2}{(a+3)^3}-\frac{c^3}{(a+4)^4}+\dots,$$

from which we have $$\int_0^1 x^{1+cx}\,.\,dx=\frac{1}{2}-\frac{c}{3^2}+\frac{c^2}{4^3}-\frac{c^3}{5^4}+\dots \qquad (2)$$

Again, writing down the expansion of x^{cx^a}, and proceeding as before, we shall obtain

$$\int_0^1 x^{cx^a}\,.\,dx=1-\frac{c}{(a+1)^2}+\frac{c^2}{(2a+1)^3}-\frac{c^3}{(3a+1)^4}+\dots,$$

from which we deduce

$$\int_0^1 x^{cx^2}.\,dx = 1 - \frac{c}{3^2} + \frac{c^2}{5^3} - \frac{c^3}{7^4} + \ldots, \qquad \text{.........................(3)}$$

while, further, we get

$$\int_0^1 x^{2+cx^2}.\,dx = \frac{1}{3} - \frac{c}{5^2} + \frac{c^2}{7^3} - \frac{c^3}{9^4} + \ldots. \qquad \text{.........................(4)}$$

In the above four forms of series, putting $c=1$, the numerators are all equal to unity; we may also express as definite integrals series with the same denominators but where the numerators may be in succession 1, 2, 3, ..., 1, 3, 5, ... For example, by using (1) and (2) we shall find the series

$$\frac{1}{2^2} \pm \frac{2}{3^3} + \frac{3}{4^4} \pm \frac{4}{5^5} + \ldots$$

to be equal to $\quad 1 + \int_0^1 (x-1)x^{-x}.\,dx \quad$ and $\quad \int_0^1 (x+1)x^x.\,dx - 1$

respectively; and in like manner obtain expressions for the series

$$1 \pm \frac{3}{2^2} + \frac{5}{3^3} \pm \frac{7}{4^4} + \ldots;$$

while, further, by using (3) and (4) we may deduce corresponding values for the series

$$1 \pm \frac{2}{3^2} + \frac{3}{5^3} \pm \frac{4}{7^4} + \ldots.$$

If we take the more general form of expression x^{x^x}, and write down its expansion by the exponential theorem, the general term of the series is $(x^x \log x)^n/n!$. To find the integral of this expression, multiplying by $(\log x)^n$ the series which is the expansion of x^{cx}, we shall get

$$\frac{1}{(-1)^n n!}\int_0^1 x^{cx}(\log x)^n.\,dx$$

$$= \frac{1}{1^{n+1}} - c\cdot\frac{n+1}{2^{n+2}} + \frac{c^2}{2!}\cdot\frac{(n+1)(n+2)}{3^{n+3}} - \frac{c^3}{3!}\cdot\frac{(n+1)(n+2)(n+3)}{4^{n+4}} + \ldots,$$

and putting $c=n$, we have the corresponding series for

$$\frac{1}{n!}\int_0^1 (x^x \log x)^n.\,dx.$$

We thus, by putting $n=0, 1, 2, 3, \ldots$, can obtain the value in series form of each term in the general integral

$$\int_0^1 x^{x^x}.\,dx.$$

Queen's Coll. Cork. A. H. Anglin.

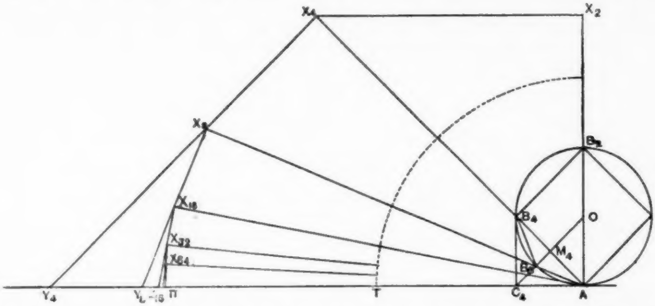
309. [K. 21. d.]. *A Geometrical Construction for* π.

There are many experimental methods of finding π, such as wrapping cotton round cylinders or rolling inky pennies on paper. When the circumference of a circle drawn on paper has to be estimated it is usual to step out the circumference with dividers or with transparent paper and a pricker. The multiplication of errors is however inevitable, and no more than two-figure accuracy can be expected in such an experiment. An examination of the well-known Vieta-Euler product formula in the shape

$$\pi = 2\sec\pi/2^2 \sec\pi/2^3 \sec\pi/2^4 \ldots$$

shews that it lends itself to geometrical construction, and further, that it has a simple interpretation. On taking unity as the diameter of a circle it is seen that $2\sec\pi/2^2$ represents the perimeter of an inscribed square, $2\sec\pi/2^2\sec\pi/2^3$ that of an inscribed octagon and so on: the limit of the product is naturally the limiting perimeter, *i.e.* the circumference of the circle. Using these ideas we may proceed as follows:

In a circle draw regular polygons of 4 and 8 sides, both starting from the point A. Let AB_4, AB_8 be the first side in each case. If AB_4 be produced to X_4 so that $AX_4=4AB_4$, then AX_4 represents the perimeter of the square.



In the same way $AX_8=8AB_8$ is the perimeter of the octagon. Now if M_4 is the middle point of AB_4, then B_8M_4 is perpendicular to AB_4 and $8AM_4=AX_4$. Hence X_8X_4 is perpendicular to AX_4. Again, if AT is the tangent at A we see that $B_8AB_4=B_8B_4A=B_8AT$, so that X_8A is the bisector of the angle B_4AT. Thus X_8 could be found by the construction: bisect X_4AT and draw from X_4 a perpendicular to X_4A. By the same method X_{16}, X_{32} ... can be determined. These points X are eventually indistinguishable from a point Π on AT, and the length $A\Pi$ must equal the circumference of the circle. It is found that the length AX_{64} does not differ appreciably from $A\Pi$. With this construction it is easy to get $\pi=3{\cdot}14\pm{\cdot}02$. The perimeters of the circumscribed polygons may be shewn in the same diagram. They are the intercepts between A and the points Y_4, Y_8 ..., where X_4X_8, X_8X_{16} ... cut AT.

The points X lie on a spiral $r=\pi \sin\theta/\theta$, which has the peculiar property that if the radius vector at one end of a chord bisects the vectorial angle at the other end, then the chord is perpendicular to the radius at the latter end. The construction for π could be varied by starting at any other point of this spiral, *e.g.*, at X_3 or X_5 instead of at X_4, just as Euler's product formula can be used in various forms for the numerical calculation of π.

Finally, it may be worth noticing that the areas of the inscribed and circumscribed polygons are $\frac{1}{2}OM\,.\,AX$ and $\frac{1}{2}OA\,.\,AY$ respectively, so that the area of the circle is $\frac{1}{2}OA\,.\,A\Pi$. F. J. W. WHIPPLE.

QUERY.

(68) The circles whose diameters are the three diagonals of a quadrilateral are known to be coaxal. By what property of the four lines which form the quadrilateral is it possible to distinguish the cases when the common points of these circles are real or imaginary?

[The other circles of the coaxal system are director circles of conics which touch the four lines; hence the limiting points (or point-circles) of the system are the centres of the two R.H.'s which touch the four lines. The common points are therefore imaginary when the two R.H.'s are real, and *vice-versa*. But I cannot obtain any simple way of expressing this condition; can any reader of the *Gazette* help me? It appears that the common points coincide if one side of the quadrilateral passes through the in-centre of its diagonal triangle. H. W. RICHMOND.]

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ERRATA.

Vol. V. p. 164, line 5, *for* 53·7′ *read* 53° 7′.
p. 166, line 14, *for* 306 *read* 307.
p. 168, line 11 up, *for* 3979 *read* 8979.

Members of the Association will regret to learn that Miss Greene, who has been appointed Inspector of Secondary Schools under the Board of Education, resigns her position as Joint Honorary Secretary of the Association.

BOOKS, ETC., RECEIVED.

Mathesis. July-August, 1909. Edited by P. Mansion and J. Neuberg. 9 frcs. per ann. (Gauthier-Villars.)

Sur les quartiques binodales quadrillées. Cl. Servais. *Sur l'Équation d'un Espace à n dimensions en coordonnées-distances.* P. Mansion. *A propos de la formule de Machin.* P. M.

Supplemento al Periodico di Matematica. Edited by Prof. Lazzeri. 2.50 l. per ann. (Giusti Livorno.)

Criteria of Divisibility. T. Ghezzi. *An Identity.* E. N. Barisien. *Solution of the equation* $x^2+y^2=z^n$. F. Ferrari.

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On the Equation $c^2=a^2+b^2$. Prof. L. Crawford. *New Clock Problems.* J. Jackson. *Multiplication and Division of Vulgar Fractions.* G. Philip. *A First Lesson in Algebra.* W. Anderson. *Geometrical Illustrations of Algebraic Identities.* P. Comrie. *Similar Figures.* P. Pinkerton. *In- and Ex-bisectors, and an Example of Continuity.* P. Pinkerton.

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Examples and Homework in Preliminary Practical Mathematics. By T. I. COWLISHAW. Part I. Pp. 81. (2nd edition.) 1s. 6d. Paper covers. Part II. Pp. 99. 1s. 6d. Paper covers. 1909. (Longmans, Green & Co.)

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Sopra i poliedri regolari convessi (concluded). E. CIANI. *Costruzione di enti immaginari* (concluded). P. BENEDETTI. *Risoluzione in numeri interi dell' equazione indeterminata,* $x^2+axy+bz^2=z^n$. F. FERRARI. *Un teorema sulle successioni di funzioni discontinue.* F. SIBIRANI, etc.

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First Meeting of the London Branch of the Mathematical Association. *29-1.10.*

Some diagrams in continuation of those presented by Mr. R. F. DAVIS at the preliminary meeting.

A B_1 C is an acute angled triangle, sides a, b, c, angle C and A B_2 C is an obtuse angled triangle sides a_2, b, c, angle 180°—C

Fig. I. consists of two rhombi, side a + b, angles C and 180°—C.

Fig. II. consists of two parallelograms, sides a + b and a_2 + b, angles C and 180°—C.

Removing the coloured areas the remainders are equal—

so

$$\left.\begin{aligned} c^2 + ax &= a^2 + b^2 \\ \text{or} \quad c^2 &= a^2 + b^2 - 2a\ CM \end{aligned}\right\} \text{I}$$

$$\left.\begin{aligned} c^2 &= a_2^2 + b^2 + a_2 x \\ &= a_2^2 + b^2 + 2\,a_2\ CM \end{aligned}\right\} \text{II}$$

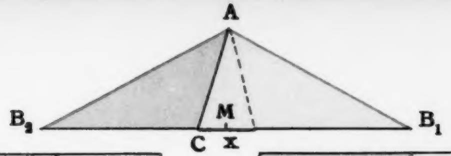
Fig. II shows also how to inscribe in a parellelogram a rhombus whose angles are equal to the angles of the parallelogram (when possible).

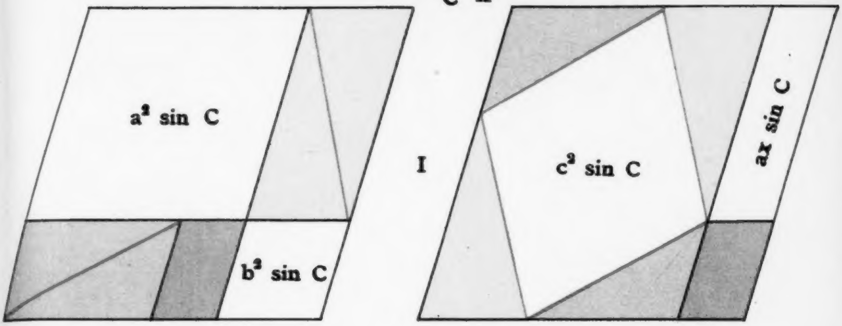
Fig. III shows that the area of the parallelogram 1 2 3 4 is equal to the difference of the rectangles A2 and A4

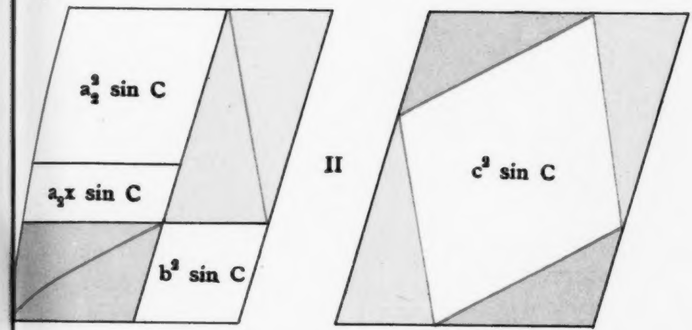
so

Area of parallelogram $= (x_3 - x_2)(y_2 - y_1) - (x_3 - x_4)(y_4 - y_1)$

C. S. Jackson.







$x = a - a_2 = 2\,\mathrm{CM}$